Algebraic Thinking in the Numeracy Project: Year One of a Three-year Study

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The Numeracy Development Project (NDP) aims to help students develop the ability to split numbers in the most suitable way in order to carry out operations mentally. In doing this, they are using numbers as quasi-variables. This study will follow a cohort of students from years 7 or 8 through to 10 to note the ability of these students to move from using numbers as quasi-variables to using letters as variables. This paper reports on the first year of this study, which included a year 7 cohort. More year 7 students generalised from numerical to literal variables than other year groups. Year 9 students, who were being introduced to formal algebra, did the most poorly in generalising from numerical to literal variables.

Students who can think of numbers as made up of parts learn that these numbers can be broken up in a variety of ways. For example, 351 is made up of seven 50s and a 1, of 345 and 6, of 360 – 9, or of 350.7 + 0.3. When the students operate with these numbers, they can use whatever division into parts makes the operation easiest to perform. Students who apply these and other operational strategies to solve problems sensibly show an awareness of the relationship of the numbers involved in the problem. They see that numbers can be quasi-variables. In our view, they demonstrate that the strategy is generalisable and so are engaging in algebraic thinking. This connection between an awareness of generality in any mathematical domain and algebraic thinking is well supported by the views of Fujii (2003), Fujii and Stephens (2001), Kaput and Blanton (2001), Lee (2001), Mason (1996), and Steffe (2001). Fujii (2003) and Fujii and Stephens (2001) extend this link between number and algebraic thinking by arguing further that, within the strategies that students devise as above and in which generality of thinking is illustrated, the numbers themselves act as variables. They refer to these numbers as quasi-variables, which Fujii elaborates as:

> a number sentence or group of number sentences that indicate an underlying mathematical relationship which remains true whatever the numbers used are. (p. 59)

In 2003, we carried out a study to examine whether students in the NDP could generalise the use of quasi-variables, using whole numbers, more successfully than comparable students who were not in the project. They could (Irwin & Britt, 2005). In 2004, we examined whether or not different groups of students could demonstrate this algebraic thinking with decimals as well as with whole numbers. They could (Irwin & Britt, 2004). Again, those students who had been in the NDP were more successful than those who had not been in the NDP.

We are now investigating whether or not students who have had the NDP in intermediate school can demonstrate this algebraic understanding after they have moved to secondary school, where the usual manner of teaching algebra is different in that it does not build on students' understanding of using numbers as quasi-variables. We intend to follow individual students' responses to the same test items over three or four years to follow their patterns of achievement.

As 2004 was the first of our three-year study, we cannot make any statements about the longitudinal effect for students in the NDP. However, we can compare year groups. This also includes comparing students in year 9 who came from schools where the NDP was in use with students from schools that were not involved in the project.

Method

Participants

Students came from four intermediate schools and the secondary schools to which most of those students would go. All the intermediate schools had participated in the NDP. Two of the pairs of schools were in Wellington, and two of the pairs of schools were in Auckland. They were chosen because of the relatively close match of the decile ranking of the intermediate and the secondary schools that most students would attend. The ethnic composition of students at these schools is shown in Table 1.

Table 1
Characteristics of Schools in the Three-year Study of Algebraic Thinking

School Type	Decile ranking	Student roll	Asian	Māori	New Zealand European	Other	Pasifika	Date of ethnicity data
Intermediate	2	216	3%	30%	46%	-	21%	11.04
	3	528	8%	28%	28%	12%	24%	11.03
	5	628	-	17%	66%	15%	2%	6.02
	6	330	3%	15%	73%	5%	4%	5.03
Secondary	3	795	4%	29%	58%	-	9%	7.02
	4	1435	11%	23%	45%	-	21%	5.04
	5	1493	6%	14%	71%	6%	3%	11.04
	7*	1253	3%	18%	73%	2%	4%	8.02

*no tests given in 2004

For reasons that suited the schools, three intermediate schools gave the test to three or four selected classes, usually selected by the willingness of the teachers to participate. The fourth school gave the test to all of their classes. Since the classes in this school mixed year 7 and year 8 students, 98 year 7 students were assessed. One secondary school chose not to participate in 2004, but it is expected that they will in 2005 and 2006. Three secondary schools gave the test to their year 9 students, and two schools gave it to their year 10 students. Details of the sample are presented in Table 2.

Table 2
Number of Schools and Students Participating in Year 1 of the Three-year Study of Algebraic Thinking, with The Decile of the School

	Number of schools	Decile of the school	Number of students participating
7	1	2	98
8	4	2, 3, 5, 6	317
9	3	3, 4, 6	781
10	2	4, 6	549

Materials

The same test was given to all students. There were five similar items requiring compensation for the four arithmetic operations: addition, multiplication, subtraction, and division. Two exemplars were provided for each of these sections. For subtraction, students were to use *Kate's method*, illustrated with 37 – 18 being transformed into 39 – 20 and 71 – 43 being transformed into 68 – 40. The items for the students were similar to: 181 – 48, 16.1 – 5.2, $48 - d = 50 - \square$, $f - 9.9 = \square - 10$[3], and $a - b = \square - (b + c)$. The first item in each section involved whole numbers, the second item included decimal fractions, the third item involved whole numbers and one literal symbol, and the fourth item included one literal symbol and a decimal fraction. The fifth item required students to complete an algebraic identity with literal symbols only.

Method

The teachers administered the test towards the end of the term 4 in normal class time on a day that suited them. Students were instructed to read the section with the two exemplars carefully, to write the answer in the space below each question, and not to use a calculator. Graduate students, who had just completed their pre-service secondary mathematics teacher education programmes, marked the tests under the guidance of the authors. Responses were credited as correct if they followed the structure of the exemplars.

Results

Tables 3–5 and Figures 1–4 show the percentage of students in each year group that solved each item in the required manner. We were particularly interested in students' ability to use a technique on items that included letters that they had previously used only on items with numerals.

Table 3
Overall Percentage of Items Completed Accurately by Each Year Group

Year	Mean score	Modal score	Percentage of students with some literal items correct
7	4.95	1	46%
8	5.2	3	26%
9	3.9	0	18%
10	5.7	0	31%

We were surprised at the marked difference between year groups in the percentage of students with some literal items correct. These data are also shown in Figure 5.

A higher percentage of year 7 students were successful on these items than were other year groups, an issue that will be discussed later. The pattern of increasing difficulty within each page and operation that this year group demonstrated is the same for all year groups.

[3] See end note.

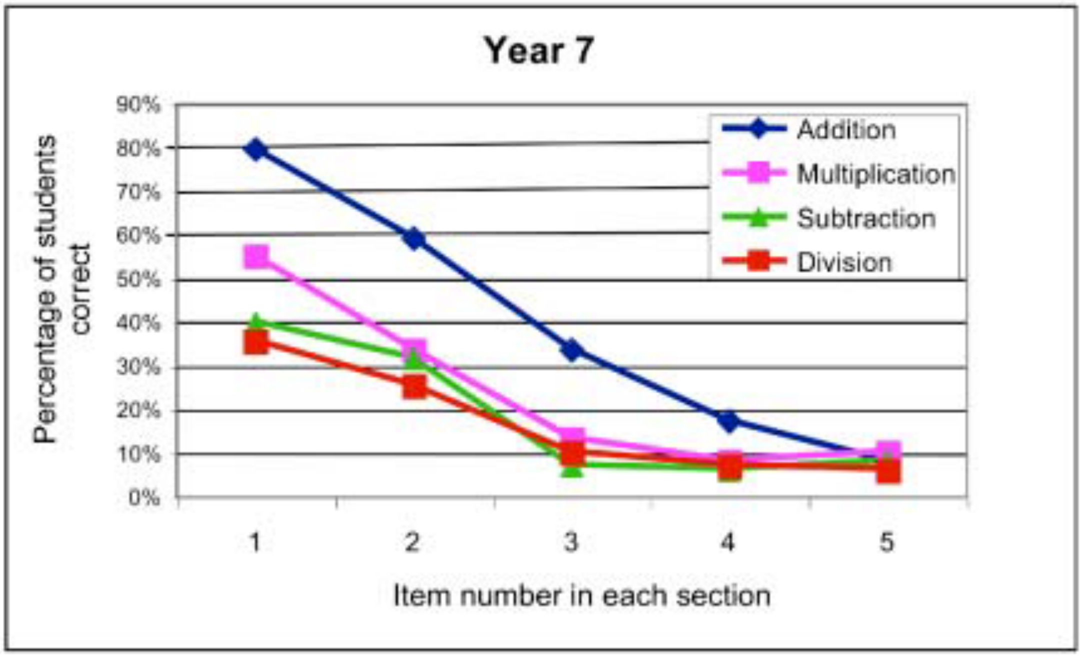

Figure 1. Results of year 7 students on the test of algebraic thinking (1 school, 98 students)

For year 7, unlike other year groups, division was the most difficult operation.

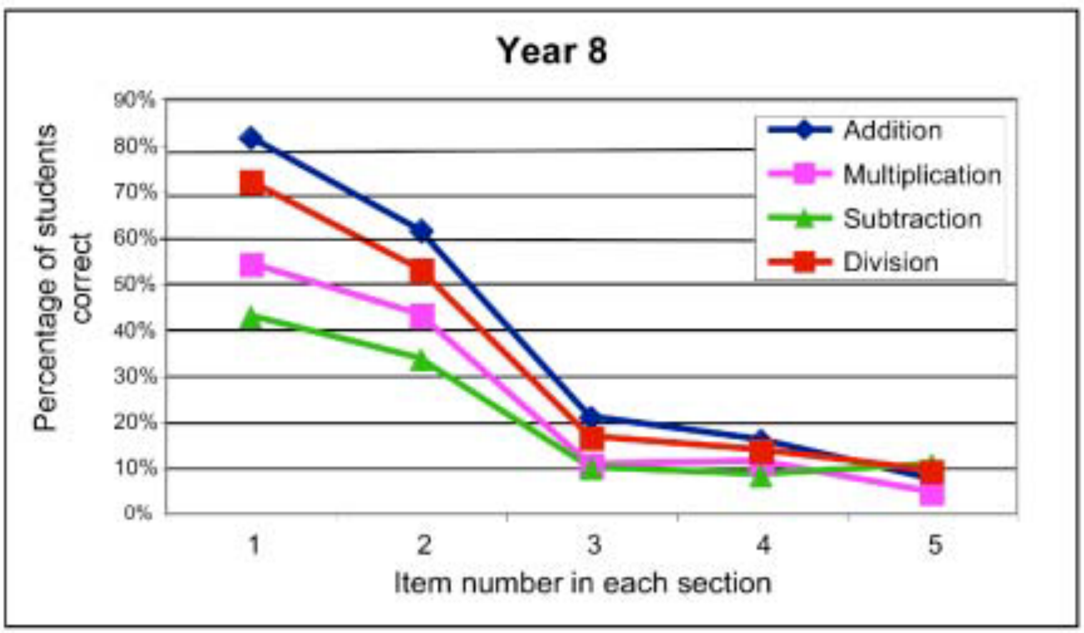

Figure 2. Results of year 8 students on the test of algebraic thinking (4 schools, 317 students)

The graph of the percentage of year 8 students who succeeded on these items showed a much sharper decline between item 2 and item 3 in addition than did the graph for year 7 students. It also showed subtraction to be the operation on which fewest students succeeded. Table 4 compares results for the four intermediate schools. As three of the schools chose which students to include, these data may not represent the whole school, except in the case of the decile 2 school.

Table 4.
Scores of Year 8 Students from Four Intermediate Schools

Decile ranking	Number of students	Mean score	Modal score	Percentage of students with some literal items correct
2	82	4.96	1	46%
3	66	4.68	0	37%
5	76	5.66	0	38%
6	93	5.61	3 and 6	19%

The differences between year 8 groups will not be important in future years of the study as each student will be compared against his or her own score in later years. However, there is some interest in the fact that the school with the lowest decile ranking, which did not select students, had a higher average score than any of the other schools in the percentage of students who were correct on some items that included letters. The students from this school appear to have done a better job at this than did the selected classes from other schools. Also, the school with the highest decile ranking had higher modal scores but few students who transferred this understanding of using numbers as quasi-variables to the use of letters as variables.

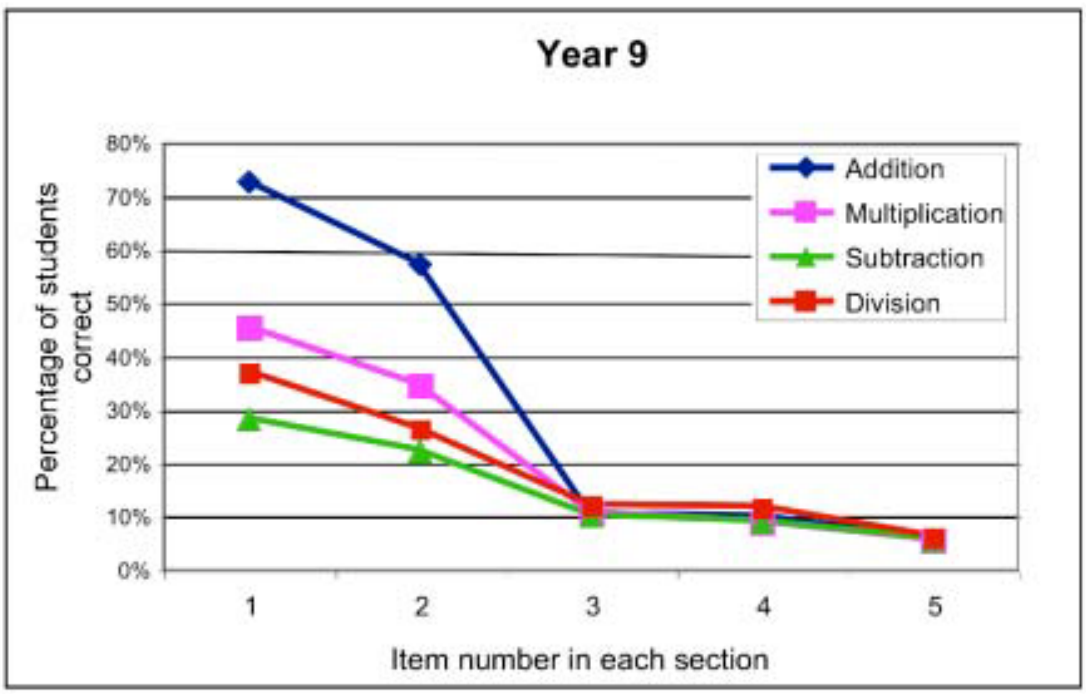

Figure 3. Results of year 9 students on the test of algebraic thinking (3 schools, 781 students)

Year 9 students did more poorly on average than did any other year group. We believe that this may relate to lack of continuity in the teaching of algebraic thinking between intermediate and secondary school.

We compared the scores of year 9 students who had attended intermediates that were using the NDP with those who had not, and there was no appreciable difference. Both groups had a mean score of 4.3 (NDP 4.31 and non-NDP 4.28). Both groups found addition to be the easiest operation and subtraction to be the most difficult. The main differences were on the first item in each section, where the students from NDP schools performed slightly better than those from non-NDP schools (see Table 5).

Table 5

Percentage of Year 9 Students Correct on the Initial Item for Addition, Multiplication, Subtraction and Division Who Had Attended NDP Intermediate Schools and Those Who Had Attended Non-NDP Schools

NDP participation	Number of students	Addition	Multiplication	Subtraction	Division
From NDP intermediates	402	75%	45%	29%	36%
From non-NDP intermediates	310	70%	45%	27%	35%

It is safe to assume that the small difference on the initial items in addition, subtraction, and division was due to some students having remembered doing items like these in the NDP. However, they failed to see the relationship of letter-based secondary school algebra to this use of quasi-variables.

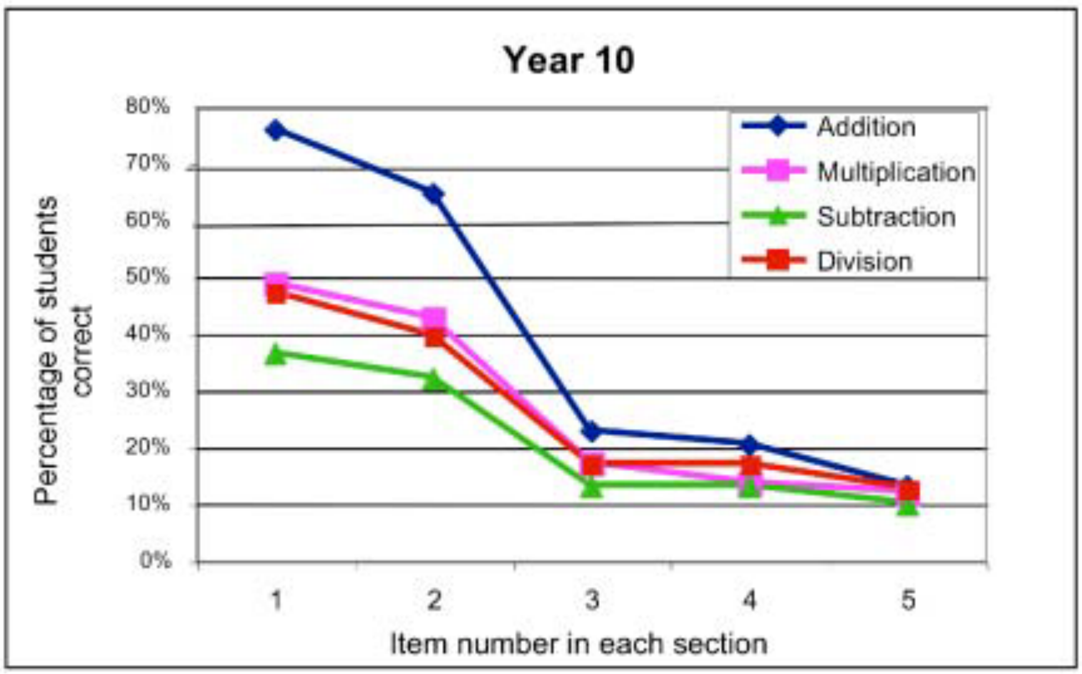

Figure 4. Results of year 10 students on the test of algebraic thinking (2 schools, 549 students)

Year 10 students did somewhat better than year 9 students on this assessment, but the pattern of achievement was similar to that for other year groups.

Our particular interest was in students' ability to generalise algebraic thinking from numerical items, something that they may have learned in the NDP, to expressing this algebraic thinking with letters as variables. Therefore we analysed the students who were successful on some numerical items and also on some literal items (see Figure 5). These figures also appear in Table 3.

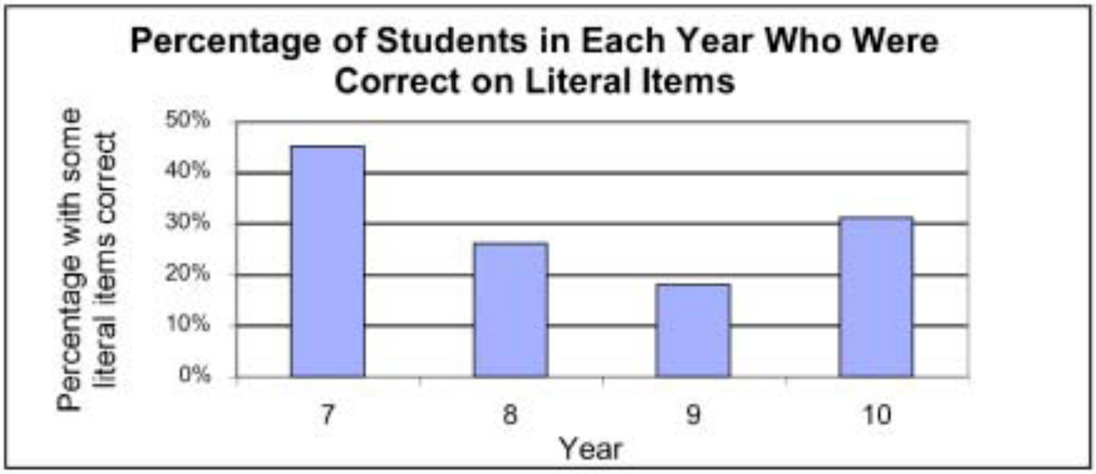

Figure 5. Percentage of students from all years who were correct on both numerical and literal items

This table shows most startlingly the superiority of the year 7 students in moving from the use of numerals as quasi-variables to using letters as variables. The year 9 students did most poorly in making this move. These year 9 students were being taught algebra in a conventional manner. This figure provides an excellent base line for determining if the new Secondary Numeracy Project will help students build new algebraic skills on their existing ones.

Discussion

In this discussion, we focus on two issues. One issue involves the students from all classes who appear to be in transition, that is, thinking algebraically on numerical items and beginning to transfer this algebraic thinking to literal items. The other issue we discuss involves possible reasons for the superiority of the year 7 group.

Students could complete up to 8 of 20 items correctly if they used algebraic thinking with numerals only. This did happen in the decile 6 intermediate school. In the decile 5 intermediate, only two students who scored 8 or less were correct on at least one literal item. However, in the decile 2 school, 19 students who scored 8 or less had some literal items correct. In the decile 3 school, 17 students who scored 8 or less had some literal items correct. These children were actively thinking in an algebraic manner. We do not know exactly what teaching occurred in their classes to encourage this thinking, but it would be worth exploring and fostering. Similarly, it would be useful to explore the teaching in schools where students could use numerals as quasi-variables but could not transfer this thinking to the use of letters. Most of the students who were accurate on some literal items scored at least a total of 5. We nominated students who scored a total from 5 to 15 as being transitional in the development of algebraic thinking. Those scoring from 16 to 20 were experts. There were 18 year 8 students scoring in this expert range.

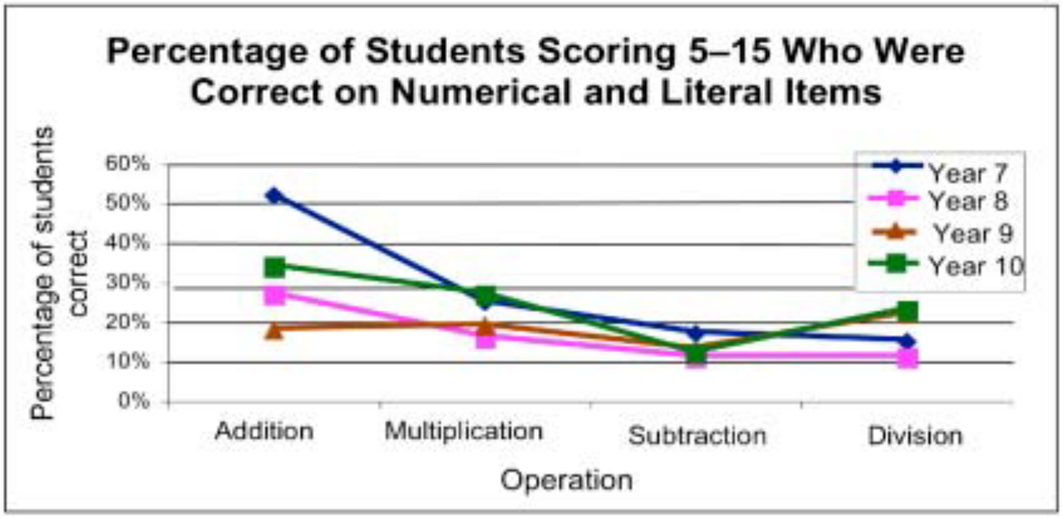

Figure 6. Percentage of students from all years who were correct on both numerical and literal items, given by operation

This figure demonstrates that addition was the easiest operation for transferring algebraic thinking from numbers to letters as variables in years 7, 8, and 10, but not for year 9, where success on literal items was generally low. It raises the question of whether or not year 8 students spend time on the use of quasi-variables in addition at the expense of the other operations. It again demonstrates the superiority of year 7 and year 10 students on this assessment of algebraic thinking.

These transitional students would appear to be those who need the opportunity to formulate the generalisation from operating with numbers as quasi-variables to expressing these operations with variables. Understanding variables, rather than unknowns, has long been difficult in secondary school algebra (Küchemann, 1981). These results suggest that students who score in this range are ready to express algebraic relationships as variables. The 19 experts found in this sample of year 8 students are already comfortable with this use of letters.

Why were the year 7 students so good? Again, we can only speculate. The fact that they were better than the year 8 students in their own school is another intriguing question. The facilitator for this school was asked for his views on this. He reports that when working with the teachers in this school he used the term "variable" from an early stage, representing it first with an empty square and then with a letter. He ran a workshop on developing algebraic thinking from number in the third term of that year and used examples from the second author in his workshops. Thus, the teachers may have introduced the term variable and the concept of moving from numbers to letters as variables in their classes. The year 7 students may have had a better understanding of this concept than the year 8 students in their own school because those older students, who were cross-grouped for mathematics, may have been introduced to algebra in a traditional manner that did not grow out of algebraic thinking with numbers as quasi-variables. We will watch this cohort in future years with interest and will be interested in the teaching that they receive as year 8 students.

Two of the secondary schools are involved in the Secondary Numeracy Project. This involvement will enable us to see if that project is able to avoid some of the drop in algebraic thinking noted in this year's cohort of year 9 students. It will be an intriguing ongoing study.

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Note: These items were presented in the form given. We were aware of possible confusions if students tackled these items with an algebraic eye rather than from the structural point of view represented in the exemplars. In #3, we wanted students to try to figure out the adjustment involving the second number, in this case where d becomes $d + 2$. In #4 by contrast, the compensation adjustment involves the first number so that f becomes $f + 0.1$. We monitored this closely during extensive trialling in full classes from years 8, 9, and 10 in different schools and found that no difficulties arose as a result of the presentation. We also talked with the students involved in the trialling about their difficulties/misunderstandings, specifically in relation to these items. The crucial point here is that the tasks were related to generalising from the numerical examples, not recalling algebraic rules when dealing with a negative in front of an expression in parentheses. There was no evidence that students gave d – 2 in the empty box arising from an expansion of $-(d + 2)$. The graphical data for items 3, 4, and 5 in the full set of data across all three operations and year groups also shows, as reported, that these items were consistently poorly done and that #3 for subtraction is not idiosyncratic when compared with the others.