Algebraic Thinking in the Numeracy Project: Year Two of a Three-year Study

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This three-year study traces the development of algebraic thinking in students who have been in the Numeracy Development Projects. We first assessed a cohort of students in 2004 when this group was in year 8. We assessed the same students again toward the end of 2005 when they were in year 9, the first year of secondary school. There was a significant increase in the scores of these students between years 8 and 9. This compares favourably with different groups of students assessed in 2004, when the year 9 students did less well than the year 8 students. However, there was also a significant difference between schools in progress made. Correlations between test scores of individual students in pairs of schools ranged from 0.06 to 0.71. Correlations on test items including letters ranged from –0.21 to 0.54. The highest correlations came from an intermediate school, whose students had the highest mean scores of all intermediate schools.

Background

In 2004, we embarked on a three-year study of the algebraic thinking of students in schools that were engaged in the Numeracy Development Projects (NDP) (Irwin & Britt, 2005a). For this assessment, we designed a test of algebraic thinking for use over the three years. This 20-item test covers the use of compensation in addition, multiplication, subtraction, and division. For each operation, the test included two numerical items, two that used a letter to express one variable, and a final item that expressed the entire concept using letters. We referred to the items that included letters as literal items as opposed to numerical items.

Results of that study in 2004 showed that some students could demonstrate understanding of the compensation principle in the example by using it in both numerical and literal items. Students in intermediate schools were more successful in demonstrating understanding of this principle than were students in secondary schools. One school had given the test to all their year 7 students as well as to year 8 students. We were surprised to see that these younger students were more successful than the older students.

This report covers the second year of this three-year study. The students who were in year 8 in 2004 were now near the end of their first year of secondary school, year 9. We will test these same students again in 2006, near the end of their year 10. Our intention is to see if these students demonstrate algebraic thinking in all three years, built on a base of what they learned in the NDP. While these students are the focus of our study, we also assessed all of the students in year 9 and year 10 in the participating secondary schools. This provides comparisons with the data in the first year of this study.

The rationale behind this study is that students in the NDP have experience in part–whole thinking that enables them to use algebraic thinking when operating with numbers. We believe that this experience in using algebraic thinking with numbers should give them an advantage when doing algebra in secondary school. We hypothesised that the Intermediate Numeracy Project (INP) was providing intermediate school students with the ability to think algebraically (see: Fujii, 2003; Fujii & Stephens, 2001; Irwin & Britt, 2005b; Kaput & Blanton, 2001; Lee, 2001; Mason, 1996; Steffe, 2001) and that this skill would enable them to succeed in algebra in secondary school.

Method

Students from one intermediate school and four secondary schools were assessed near the end of 2005. It had been our intention to assess only students in the four secondary schools that took students from the intermediate schools assessed in 2004. However, the results from one intermediate school in 2004 were so surprising that we decided to assess them again this year to see if similar results were obtained.

In the report of the 2004 data (Irwin & Britt 2005a), we identified the schools by their decile ranking. The rankings in 2005 (see Table 1) changed from the 2004 rankings due to changes in Government policy. In this 2005 report, we identify the pairs of schools by a number but also, in Table 1, give their decile ranking in 2005 and in 2004.

When these secondary schools were chosen, the nature of the NDP for secondary schools was still being formulated. We did not know if any of these secondary schools would be engaged in the NDP. It is serendipitous that three did participate in the Secondary Numeracy Project (SNP) in 2005. The school that did not participate in the SNP developed their own programme, which concentrated on numeracy and algebra throughout the year.

Participants

The characteristics of the 2005 schools and participants are shown in Table 1.

Table 1
Participants in the 2005 Cohort of the Test of Algebraic Thinking

School	Decile ranking in 2005	Decile ranking in 2004	Year group	NDP involvement	Number of students assessed	Decile of contributing intermediate school in 2004	Number of students assessed in 2004 and 2005
Intermediate 1	2	2	7	INP	80		
Intermediate 1			8	INP	93		67
Secondary 1	4	3	9	SNP	142	2	43
Secondary 1			10	SNP	153		
Secondary 2	4	3	9	SNP	237	3	21
Secondary 2			10	SNP	224		
Secondary 3	6	5	9	Own programme	338	5	69
Secondary 3			10	Own programme	322		
Secondary 4	8	7	9	SNP	260	6	52
Secondary 4			10	SNP	282		

Schools gave the test to all available students in each class. Our primary focus in 2005 was on the students who were assessed in 2004 and 2005, as shown in the last column of Table 1.

All secondary schools except secondary school 4 were participants in this study in 2004. That school had been contacted in 2004, but the head of the mathematics department was leaving and did not want to commit the next head of department to a three-year study. The new head of mathematics was happy to participate.

In addition to assessing these students, we interviewed teachers in the mathematics department of the four secondary schools about the usual ways in which they taught algebra, their knowledge of the NDP, differences they noted between students from NDP schools and those from non-NDP schools, and related topics. We also spoke with the facilitator, principal, and deputy principal of the one intermediate school about possible reasons for the success of their students in 2004.

Materials

The same test was given to all students. This was identical to the test given in 2004. There were five items requiring compensation for the four arithmetic operations: addition, multiplication, subtraction, and division. Two exemplars were provided for each of these sections. For addition, students were to use *Jason's method,* illustrated by 27 + 15 being transformed into 30 + 12 and 34 + 19 being transformed into 33 + 20. The items for the students were similar to: 198 + 57, 25.7 + 9.8, $48 + n = 50 - \square$, $8.9 + k = 9 + \square$, and $a + b = (a + c) + \square$. The first item in each section involved whole numbers, the second item included decimal fractions, the third item involved whole numbers and one literal symbol, and the fourth item included one literal symbol and a decimal fraction. The fifth item required students to complete an algebraic identity with literal symbols only.

Procedure

The teachers administered the test towards the end of term 3 or early in term 4 in normal class time on a day that suited them. Students were instructed to read the section with the two exemplars carefully, to write the answer in the space below each question, and not to use a calculator. Graduate students, who had just completed their pre-service secondary mathematics teacher education programmes, marked the tests under the guidance of the authors. Their marking was checked by the second author and re-marked where necessary. Responses were credited as correct if they followed the structure of the exemplars.

Results

The main focus of our analysis was on the students who had taken this test the previous year, as stated above. However, the performance of all students was analysed to see how overall results compared to that of students in the previous year.

Results for All Students

Table 2 gives the percentage of all year 9 students that were successful on this test in 2005, with the percentage of year 9 students correct in 2004 given in brackets. The 2005 percentages for all items for all year groups is given in Appendix L.

Table 2
Percentage of Year 9 Students Correct on the Test of Algebraic Thinking on Each Item in 2005 (percentage correct in 2004 in brackets)

Operation	Item 1	Item 2	Item 3	Item 4	Item 5
Addition	68 (73)	57 (57)	16 (10)	15 (10)	7 (6)
Multiplication	43 (45)	32 (35)	13 (11)	12 (9)	6 (5)
Subtraction	26 (28)	21 (22)	12 (10)	10 (9)	7 (5)
Division	34 (37)	29 (26)	14 (12)	13 (12)	7 (6)

This table shows that more students were correct on the initial item for each operation in 2004. Literal items (items 3–5) were answered correctly by somewhat higher percentages in 2005. Numerical items continued to be answered correctly by more students than were literal items. Despite the differences between 2004 and 2005, the overall pattern of results was similar for the two years. Compensation in addition continued to be the easiest operation, while compensation in subtraction was the most difficult.

Figure 1 gives a comparison of students' percentages correct on literal items only, both in 2004 and in 2005. These items were judged to give the best indication of understanding of the principles involved, demonstrating algebraic thinking. The irregular pattern of successes in 2004 had been surprising. They led us to speculate that the year 7 students were better prepared in algebraic thinking than were older students. This irregular pattern of success was not repeated in 2005. Note that in the data presented in Figure 1, the 2005 data for year 7 students comes from only one school, the same school as in the year 7 data for 2004. The data for year 8 in 2005 comes from one school, in comparison with four schools in 2004. Year 9 students come from four schools in 2005 (three in 2004), and year 10 data from four schools (two in 2004).

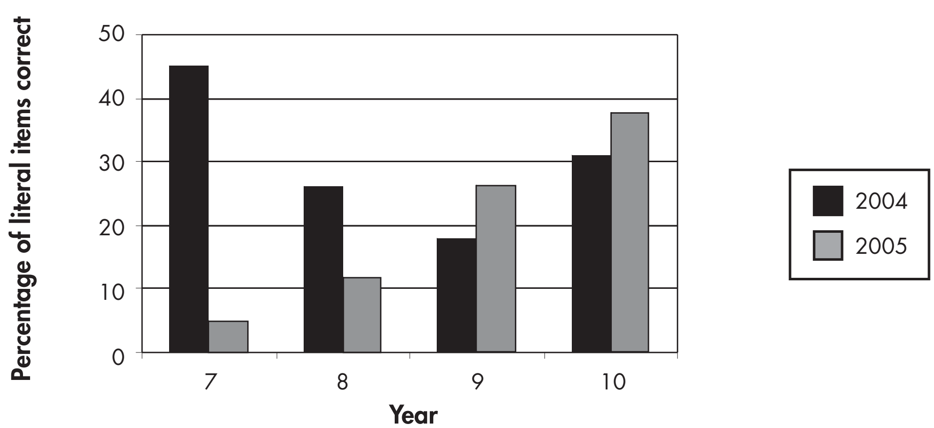

Figure 1. Percentage of students who were correct on some literal items in 2004 and 2005

Clearly, the year 7 cohort in 2005 did much less well than the year 7 cohort in 2004. It is also interesting that the year 9 students did better in 2005 than in 2004, as noted in Table 2. Little can be said about the fact that they did better than the year 8 students because only one school is represented in the year 8 data.

We looked more closely at the results from the one intermediate school assessed this year, comparing it with the results from the same school in 2004. This comparison is shown in Table 3.

Table 3
Results for Year 7 and Year 8 Students from Intermediate School 1 in 2004 and 2005

Year	Number of students in year 7	Year 7 mean	Percentage of students in year 7 correct on at least one literal item	Number of students in year 8	Year 8 mean	Percentage of students in year 8 correct on at least one literal item
2004	98	4.96	45	82	4.55	37
2005	80	2.36	5	93	3.70	12
Same students in 2004 and 2005	67	5.10	43	67	3.75	19

This table shows that while 45% of year 7 students answered literal items correctly in 2004, only 5% of the year 7 students did so in 2005. The same 67 students did better, on average, in year 7 in 2004 than they did a year later. Twenty-one of 29 students who had been able to generalise from numerical to literal items in 2004 did not do so in 2005. We asked the principal, acting principal, and facilitator if they had any idea why the year 7 cohort in particular did well on this test in 2004, but they had no clear idea. They did suggest that teachers had benefited from a workshop on algebra in 2004 and informed us that the school had changed its focus to reading in 2005.

It seems likely that some recent teaching may have been responsible for the good performance of the year 7 group in 2004 but that the effect of this teaching did not stay with the students and was not repeated for the 2005 year 7 cohort.

Results for the Students who Took the Test in 2004 and 2005

As said above, our primary interest was in the students who had been in year 8 in 2004 and had now moved to year 9 in secondary schools. Overall, these students did significantly better in year 9 than in year 8 ($F = 10.286$ [$df = 1, 179$], $p < 0.01$). However, there was also a significant difference between schools ($F = 5.370$ [$df = 3, 179$] $p < 0.01$).[1] Table 4 presents the data on students who had taken this test near the end of year 8 and near the end of year 9.

[1] Because the distribution of scores was skewed to the right, analysis was carried out on the square root of the scores.

Table 4
Results for the Same Students in Year 8 in 2004 and Year 9 in 2005

Pair of intermediate and secondary schools	Number of students assessed twice	Year 8 mean	Percentage of students in year 8 correct on at least one literal item	Year 9 mean	Percentage of students in year 9 correct on at least one literal item	Correlation of total scores in year 8 and year 9 for the same students
1	43	4.45	40	3.38	10	0.06
2	20	4.25	25	6.10	4	0.53
3	69	6.04	22	8.20	50	0.71
4	52	4.71	06	7.27	42	0.54

Table 4 shows that school pair 1 had a low correlation between scores in year 8 and year 9, while pair 3 had a high correlation.

Figure 2 presents the mean scores with standard error of measurement for the four pairs of schools graphically. The pair of schools numbered 3 includes the only secondary school not in the SNP. This school and its contributing intermediate had the highest proportion of successful students.

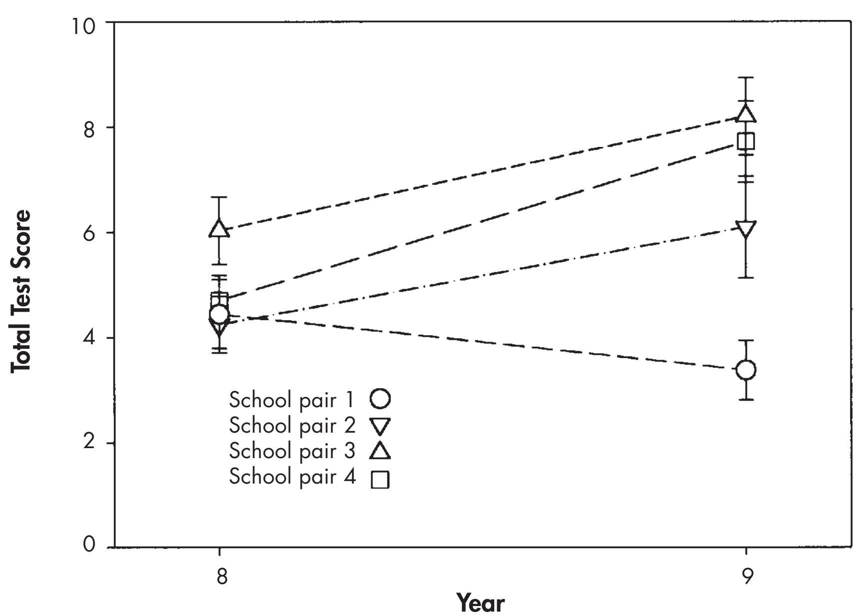

Figure 2. Mean scores and standard errors of the full test for the same students in four pairs of schools

This figure demonstrates that the students from school pair 1 did more poorly in year 9 than in year 8, while the means for students in other pairs of schools increased.

Figure 3 shows the mean totals of the 12 items that involved letters in these four pairs of schools.

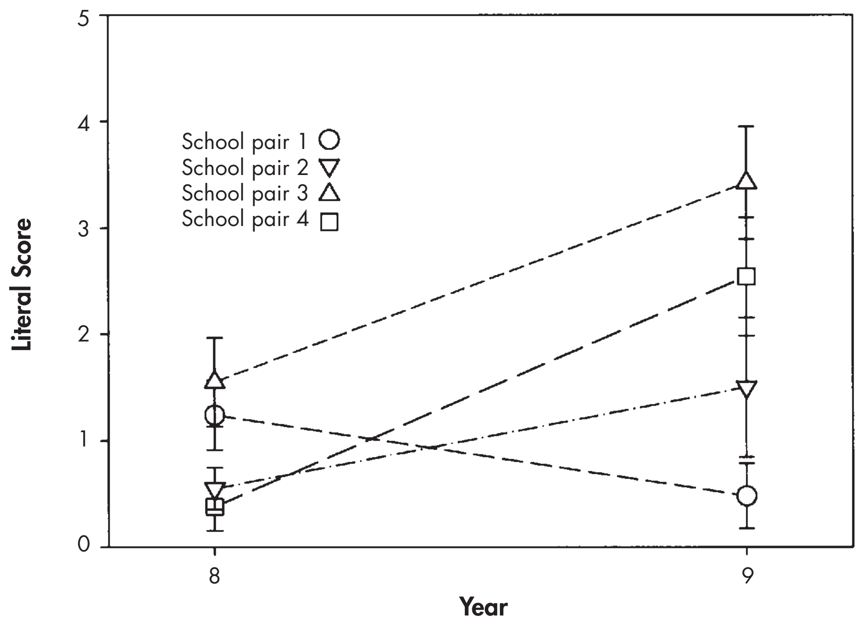

Figure 3. Mean number of items with letters in them and standard errors for the same students in four pairs of schools

This figure shows the superiority of school pair 3 on literal items. On this analysis, as for the analysis of the full scores, pair 1 decreased and the other pairs increased. The correlations for these four pairs of schools ranged between –0.21 for school pair 2 and +0.54 for school pair 3.

In 2005, the majority of year 9 students came from intermediate schools or full primaries that had entered data on the NZ Maths website (www.nzmaths.co.nz). There was no significant difference between those students and those from schools that had not entered data. This may have been because some students came from NDP schools that elected not to enter data or because many schools that do not enter their data on the national website may still have some knowledge of the NDP, possibly through the work of private providers or through reading material on the NZ Maths website.

Interviews with the secondary mathematics teachers were analysed for similarities and differences. Teachers at all schools gave similar responses when asked how they introduced algebra and what were the main difficulties that students had with this topic, although there was a variety of responses within each school. At all four schools, some teachers voluntarily mentioned the link between algebra and number or the NDP. Teachers from two schools did not believe that students from the NDP schools did any better than students from schools that had not been in the NDP (schools 3 and 4). Teachers from the other two schools found that students who had been exposed to the NDP were more open to different ways of thinking, tended to be co-operative learners, were more advanced in part–whole thinking (school 2), were more confident, had better attitudes, and were willing to take risks (school 1). However, comments from teachers did not match well with students' success on this test of algebraic thinking.

Discussion

There are only a few things we can say for certain as a result of these 2005 results. One is that the students who took the test in both 2004 and 2005 did significantly better in their second year. This is in contrast to the pattern provided by different students in years 8 and 9 in 2004. It suggests that many students are carrying forward their understanding of algebraic thinking. The other thing that we can say definitively is that there were significant differences between school pairs. This is shown most markedly by school pair 1, the one cohort of students to do more poorly in 2005 than in 2004. We can only guess at the reason for this. We understand that the intermediate students focused on algebra not long before taking this test in 2004, and we also understand that there were some changes of staffing at the secondary school that could have affected teaching there. There is not enough evidence of the relationship of these factors to the students' success to consider them to be causal.

School pair 3 were the most successful. They also had the highest correlations between student success on the two occasions. Possible reasons for this lie with both the intermediate school and the secondary school. The intermediate school, which has been involved in the NDP for 3 years, is reported to have a particularly skilful numeracy leader. Their students outscored other schools in 2004 and may have carried their algebraic thinking with them into secondary school. The secondary school was not in the SNP but chose to develop their own programme, teaching algebra throughout the year while concentrating on numeracy for the weaker students. It is possible that teachers could be more committed to a teaching programme that they had developed themselves, probably one that built on what they were already doing, than they would be to a different programme developed elsewhere. By teaching algebra for the whole year, their students had more chance to master the concepts in this test.

In this three-year study, we are looking for patterns of performance that might indicate that students who had the NDP in intermediate school carried their understanding of algebraic thinking, used primarily with numbers, with them into secondary school. We believe that this should enable them to have a basic understanding of algebra that would enable them to succeed at this most important topic.

At this point in the three-year study, results are not clear. We cannot say that all intermediate schools give their students the grounding in algebraic thinking that will give them an advantage in secondary school algebra. We have no index of quality control for teaching and learning in the intermediate or the secondary schools, nor is it possible to get one in a study of this sort. Although we cannot say so with confidence, we believe that it is likely that differences in teaching have a lot to do with the unusual results we have shown to date.

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